Induction and Confirmation Theory: An Approach based on a Paraconsistent Nonmonotonic Logic

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Abstract: This paper is an effort to realize and explore the connections that exist between nonmonotonic logic and confirmation theory. We pick up one of the most wide-spread nonmonotonic formalisms – default logic – and analyze to what extent and under what adjustments it could work as a logic of induction in the philosophical sense. By making use of this analysis, we extend default logic so as to make it able to minimally perform the task of a logic of induction, having as a result a system which we believe has interesting properties from the standpoint of theory of confirmation. It is for instance able to represent chains of inductive rules as well as to reason paraconsistently on the conclusions obtained from them. We then use this logic to represent some traditional ideas concerning confirmation theory, in particular the ones proposed by Carl Hempel in his classical paper "Studies in the Logic of Confirmation" of 1945 and the ones incorporated in the so-called abductive and hypothetico-deductive models.
Key-words: Abduction; Default logic; Hempel´s confirmation theory; Inductive inference; Paraconsistent logic

Resumo: Este artigo é uma tentativa de explorar as conexões que existem entre lógica não-monotônica e teoria da confirmação. Mais especificamente, escolhemos um dos formalismos mais utilizados em lógica não-monotônica – lógica *default* – e analisamos até que ponto e sob que condições ela poderia ser usada como uma lógica da indução no sentido filosófico do termo. Fazendo uso desta análise, nós estendemos a lógica *default* de forma a torná-la minimamente capaz de realizar o propósito da lógica indutiva, resultando em um sistema que, acreditamos, possui propriedade interessantes do ponto de vista da teoria da confirmação. Por exemplo, tal sistema é capaz de representar cadeias de regras indutivas bem como raciocinar paraconsistentemente sobre as conclusões obtidas delas. Nós então usamos esta lógica para representar algumas idéias tradicionais relacionadas com a teoria da confirmação, em especial as idéias propostas por Carl Hempel em seu artigo clássico "Studies in the Logic of

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Confirmation" de 1945 e as idéias incorporadas pelos assim chamados modelo abdutivo de confirmação e modelo hipotético-dedutivo.
Palavras-chave: Abdução; Inferência indutiva; Lógica *default*; Lógica paraconsistente; Teoria da confirmação de Hempel

1 Introduction

The study of non-deductive inferences has played a fundamental role in both artificial intelligence (AI) and philosophy of science. While in the former it has given rise to the development of *nonmonotonic logics* [14, 23, 28, 29, 30, 32], in the later it has attracted philosophers in the pursuit of a so-called *logic of induction* [10, 19, 20, 21, 34], which was supposed to formalize the logical properties of the relation of *evidential* or *inductive confirmation*. However and despite of this, perhaps because the technical devices used in these two areas were *prima facie* quite different, the obvious fact that both AI researchers and philosophers were dealing with the same, or almost the same problem in their efforts to formalize and/or clarify non truth-preserving inferences has not yet been fully explored.

This paper might be seen as an effort to effectively realize and explore this connection between nonmonotonic logic and inductive logic, broadly conceived as field and not limited to any specific formal paradigm. More specifically, we shall pick up one of the most wide-spread nonmonotonic formalisms – default logic [32] – and analyze to what extent and under what adjustments it could work as a logic of induction in a broader philosophical sense. By making use of this analysis, we shall extend default logic so as to make it able to minimally perform the task of a logic of induction, having as a result a system which we believe has interesting properties from the standpoint of theory of confirmation. It is for instance able to represent chains of inductive rules as well as to reason paraconsistently [12] on the conclusions obtained from them. We shall also use this system to represent some traditional ideas concerning confirmation theory, in particular the ones proposed by Carl Hempel in his classical paper "Studies in the Logic of confirmation" of 1945 [19] and the ones incorporated in the so-called *abductive* and *hypothetico-deductive* models of confirmation [34, 39].

The structure of the paper is as follows. In the Sections 2 and 3 we find out to what extent and under what adjustments default logic could be seen as a logic of induction. In Section 4 we make use of this analysis to introduce our modified version of default logic, being this analysis and the conclusions derived from it what justify the features of our extended default logic, including its paraconsistency. In Section 5 we apply our logic to the formalization of what we call Hempel's model of confirmation and the abductive and hypothetico-deducive models of confirmation. Finally, in Section 6, we present some concluding remarks.

2 Is Default Logic a Logic of Induction?

Since the time of Rudolf Carnap [10, 11], induction has been conceived (even though not uncontroversially) as *the class of rational non truth-preserving inferences*. The conception of non truth-preserving inference is straightforward: it simply means a non-deductive inference, that is to say, an inference whose conclusion may be false even when its premises are true. This contrasts with the second key term in the definition, the term "rational", which philosophers have shown to be a quite problematic term, both in its characterization as in its operationalization.

However, and despite of this, we might say there is a close relationship between this hard, positive side of the concept of induction and the notion of *confirmation*. According to Carnap [10] and Carl Hempel [19], for example, the purpose of the logic of induction is basically one of confirmation, i.e., given a piece of evidence *e* and a hypothesis *h*, it should say whether (and possibility to what extent) *e* confirms or gives evidential support to *h*. This is supposedly enough to distinguish between inductive inferences and other kinds of non-deductive inferences such as fallacies, for example: despite being non truth-preserving inferences, the premises of a fallacy do not evidentially support its conclusion. We shall call from now on statements of the form "*e* confirms (to degree *p*) *h*" *confirmation sentences*.

Now, what parallels can we make about this conception of induction and default logic, which, as we have said, is one of the most widespread frameworks used in AI to represent non-deductive inferences? First of all, considering single defaults as inferences rules, it is clear that default logic satisfies the negative, non truth-preserving feature of inductive inferences: conclusion β of default $\alpha{:}\beta/\beta$, for instance, may be false even in the case where its premise α is true.

Secondly, there is a strong parallel between default rules and the *qualitative form* of confirmation sentences. Since default rule $\alpha{:}\varphi/\beta$ allows us to infer β only provisionally, we can say it means something like "α

might be taken as an evidence for the hypothesis that β is the case, with the proviso that $\neg\varphi$ is not the case." Taking a well-known example, the fact that Tweety is a bird confirms or gives evidential or inductive support to the hypothesis that it flies unless we know that it does not fly. In this way, we can read $\alpha{:}\varphi/\beta$ as "α confirms or inductively supports β, with the proviso that $\neg\varphi$ is not the case," or equivalently, "α inductively supports β unless $\neg\varphi$." In order to make this reading more explicit, let us represent default rules a bit differently, writing $\alpha \succ \beta \precnsim \varphi$ to mean "α confirms or inductively supports β, unless φ." $\alpha \succ \beta \precnsim \varphi$ of course intends to mean the same as default rule $\alpha{:}\beta\wedge\neg\varphi/\beta$. Needless to say, we shall restrict our discourse to semi-normal defaults due to the obvious fact that if $\neg h$ is the case then neither e nor any other sentence can be said to confirm h; this shall automatically prevent the so-called abnormal defaults [30].

Thirdly, the rational side of inductive inferences is definitely not taken into account by default logic. While a logic of induction is supposed to function as a black box having as input several pairs of sentences *e-h* and as output a smaller number of sentences of the form "*e* inductively supports *h*," default logic deals only with the output of such box, functioning as a tool for representing such confirmation sentences and detaching their hypotheses from the evidences. As a consequence of that, it is aloof from the problem of positively characterizing inductive inferences: such a task is left to the knowledge engineer, who can use default logic to formalize rational non truth-preserving inferences but also any sort of nonsense.

Despite this, we can always wonder under what adjustments could default logic possibly work as a logic of induction. Using the black box analogy and seeing default rules as qualitative confirmation sentences, it is clear that in order to deal with both input and output of the box we need a way to 'generate' these confirmation sentences or defaults, or in other words, a mechanism to produce statements of the form $\alpha \succ \beta \precnsim \varphi$. But how are we to construct such a mechanism? A promising approach could be to use the very representational mechanism of defaults to do the job. In other words, more or less alike to nonmonotonic conditional logic [7, 14], we could allow defaults to have defaults as consequents, being able in this way to represent defaults which could have other defaults as their nonmonotonic conclusions. This would result in something like a meta-default logic, allowing us in fact to represent what we might call *calculi of defaults*. Before trying to work on this approach, let us first take a look at another feature of the philosophical project of building a logic of induction which happens to be of fundamental importance to our comparative study: the notion of probability.

If we have that evidence *e* confirms or supports hypothesis *h*, it is natural to wonder what we can conclude about *h* when *e* is true. According to the philosophical school that guided most of the attempts to develop a logic of induction – Carnap´s school of logical probability – the answer is nothing [10]. Nevertheless, there is still a strong intuition according to which if *h* itself cannot be concluded, something about it should. Thus, many philosophers have tried to find out in what circumstances we can detach the hypothesis from the evidences and conclude something about it [22, 24]. Despite the diversity of approaches, all theorists agree on one basic point: given that *e* confirms *h* and that *e* is true, whatever we conclude about *h* it should reflect the uncertainty inherent to inductive inferences.

Almost invariably some *probability* notion has been chosen to do this job: even though from "e confirms h" and "e is true" we cannot conclude that *h* is true, we can conclude that it is probable. This notion of probability should not be confounded with the object of Carnap's conceptual explanation endeavors: while the later, called by him *logical probability*, is supposed to be a purely logical notion connecting logically two sentences, the former must be seen as an epistemic label we attach to inductive conclusions in order to make explicit their defeasible character. Carnap calls this non-logical notion of probability *pragmatical probability* [11].

As far as we are concerned, it should be acknowledged that using a term like "probability", which has over half dozen established different interpretations [40], with a somehow new (or at least not widely known) meaning may be troublesome. Because of that we shall use the more qualitative and hopefully less problematic term "*plausibility*" to denote the status the conclusion of an inductive inference gets when its premises are true. It should be pointed that this choice is not a mere terminological matter. Like the concept of probability, the notion of plausibility has both the negative or defeasible aspect required by inductive inferences (which is of course related to their truth-preserving feature) as well as the positive side required by their rational character.

To be sure, on the one hand, plausible statements can be refuted: they are not certain, unquestionable, but are subject to revision and therefore can be defeated; on the other hand, we are not ready to accept a hypothesis as plausible unless there are very good reasons in its favor. In the case of inductively obtained plausible statements, these reasons may be seen as the inductive inferences themselves, or more precisely as the supposed logical rectitude these inferences posses. Because of that, a better term for our pragmatical probability would be *inductive plausibility*. Besides, the notion of plausibility has a qualitative trait (which our qualitative approach to confirmation sentences shall require) that the term probability lacks. Although Carnap, for instance, has found a place for a qualitative notion of probability in his taxonomy of probability notions [10], undoubtedly most of the uses philosophers and mathematicians have made of the term "probability" has had some quantitative bias.

Now, what implications this characterization of inductive inferences in terms of pragmatical probability or plausibility has to our previous understanding of confirmation sentences? Well, when we do not consider detaching the hypothesis from the evidences we can correctly say that the truth of *e* confirms or evidentially supports the truth of *h*. However, given that *e* evidentially supports *h*, we have seen that what the truth of *e* warrants us to infer is the plausibility of *h*, not its truth. Therefore, if we are effectively taking into consideration inferring *h* from *e* we cannot speak any more in terms of truthfulness: in this situation we have to use something alike to our inductive plausibility notion. Put otherwise, if we want to speak of something like an *inductive implication*, that is to say, an inferential version of the notion of confirmation sentence, then we should say that evidence *e* inductively implies the *plausibility* of hypothesis *h*.

Turning back now to default logic, it is pretty clear that it embodies a mechanism for detaching the hypothesis from the evidences. After all, its whole purpose is exactly to find out which consequents can obtained from a specific set of default rules and ordinary formulae. This means that besides being possibly read in terms of confirmation sentences, defaults of the form $\alpha \succsim \beta \precnsim \varphi$ can also be read in terms of inductive implications. In other words, $\alpha \succsim \beta \precnsim \varphi$, which we have agreed can be seen as meaning "α confirms or inductively supports β, unless φ" also mean "α inductively implies β unless φ".

Here however we arrive at a breakdown in our reading of default rules in terms of confirmation theory. Since original default logic provides no means to distinguish inductively or nonmonotonically obtained conclusions from deductively obtained ones, there is no way to represent "β is plausible" and consequently no way to represent statements like "α inductively implies the plausibility of β unless φ." One might be wondering whether there is some harm in that. First of all, it does harm from a philosophical point of view, by not properly representing the epistemic status of inductive statements and consequently promoting confusion between deductively established conclusions and defeasible ones. From the point of view of the inferential model, this lack of notational precision has also some problematic consequences, which are centered around the question of whether or not nonmonotonic conclusions should be treated in further reasoning in the same way as monotonic ones.

As it is well known, one of the most serious problems of non-deductive reasoning is the arising of *contradictions*. While in nonmonotonic logic in general and default logic in particular this is known as the problem of *anomalous extension* [30], in philosophical literature it has been called the problem of *inductive inconsistencies* [18]. According to some theorists, the phenomenon of appearance of inconsistencies is not simply an unfortunate feature of the available formalisms, but is an inevitable and essential characteristic of nonmonotonic reasoning in particular and inductive reasoning in general [30, 31, 37, 38]. By considering seriously this point, some have suggested to account for this problem by embodying some sort of *paraconsistency* [1, 3, 15, 25, 30], i.e., some mechanism capable of reasoning non-trivially about those inductive inconsistencies.

The whole idea of this approach is that since the support given by an inductive inference is weak, when faced with an inductive inconsistency the best attitude is to tolerate it, for new evidence may favors one of its parts and consequently dissolve the problem; on the other hand, for deductive and consequently non-defeasible conclusions, contradictions are something which shall be avoided: they should be treated classically. Note however that in order to carry this project out it is essential to distinguish between deductively established conclusions and inductive ones. Otherwise how could we reason classically upon one and paraconsistently upon the other? Therefore, following this path shall force us to make explicit the distinguishing status that inductive conclusions have, leading us thus to something very alike to our plausibility notion.

3 Towards a Logic of Inductive Implication

According to what we have discussed so far, a promising attempt to transform default logic into a logic of induction must encompass two things. First it must allow us to inferentially obtain defaults or inductive implications. Second, it shall have a way to mark the consequent of inductive implications with some plausibility symbol so as to be able to express its special epistemic status and tolerate the contradictions that may arise from the use of such implications.

For the first task we must find a way to inferentially obtain defaults or inductive implications of the form $\alpha \succ \beta \precnsim \varphi$. As we have said in the previous section, an interesting way to do that would be to allow defaults to have other defaults as their consequents. This would make possible for us to represent different calculi of defaults or, using a term borrowed from confirmation theory, different *models of confirmation*, that is to say, ways through which statements of the form $\alpha \succ \beta \precnsim \varphi$ are generated.

For the second task, we shall refer to some recent theoretical results pointing to some interesting connections between paraconsistent logic and normal modal logic [5, 27, 37] and use this latter as a sort of logic of plausibility. In [5] for instance it is shown that if we define in S5 an operator $\sim$ as $\sim\alpha =_{def} \Diamond\neg\alpha$, we have that $\sim$ can be very fairly taken as a paraconsistent negation. First, we have that it is not the case that $\alpha, \sim\alpha \vdash \beta$ for every β, and second that $\sim$ satisfies many properties classically associated to negation, such as $\alpha \vee \sim\alpha$, $\sim(\alpha \wedge \sim\alpha)$, $(\sim\alpha \rightarrow \alpha) \rightarrow \alpha$, $(\alpha \rightarrow \beta) \rightarrow \sim\alpha \vee \beta$ and $\sim\sim\alpha \rightarrow \alpha$. At the heart of this is of course the semantic structure of modal logic explored in the interpretation of $\Diamond$: since there is a model which satisfies both α and $\Diamond\neg\alpha$, they can coexist without trivializing the theory. This entitles us then to say that S5 in particular and normal modal logic in general embodies a subtler sort of paraconsistency which some have named “hertian” [5] or conceptual [37] paraconsistency.

Following then this general idea, we shall use traditional modal logic as a logic of plausibility and read $\Diamond$ as "*it is plausible that*" (instead of "it is possible that"). This logic of plausibility shall be used as the monotonic basis of our modified default logic. The practical connection between the two formalisms is that we shall force the consequents of defaults to be of the form $\Diamond\alpha$. As a result of that, what we have called inductive inconsistencies, that is to say, inconsistencies arisen from the use of inductive inference rules, will necessarily be of the form of $\{\Diamond\alpha, \Diamond\neg\alpha\}$, being quite naturally assimilated in traditional modal logic without provoking the splitting of extensions or anything alike. From now on we shall call these inductive contradictions simply *plausible contradictions*.

One might object that since $\{\Diamond\alpha, \Diamond\neg\alpha\}$ does not trivialize the theory (simply because it is not, from a technical point of view, a contradiction), our use of the terms "contradiction" and "paraconsistency" is misleading. However, from a conceptual point of view, to take something and its negation as simultaneously plausible *is* a contradiction, a contradiction internal to the domain of plausibility which is independent of the formalism we eventually decide to use. For instance, if we had used an approach in the style of da Costa's calculi [12] and represented plausible hypotheses through, say, °-less formulae, the plausible contradictions would have the shape of formal contradictions.[2]

Now that we have decided to read $\Diamond$ in terms of plausibility, we might fairly replace the notion of possible world by the notion of *plausible world*. In this way, considering a universal accessibility relation "α is plausible" (in symbols: $\Diamond\alpha$) is true iff there is a plausible world w such that α is true in w; if both α and $\neg\alpha$ are plausible (in symbols: $\Diamond\alpha$ and $\Diamond\neg\alpha$), we have that there are two plausible worlds w and w' such that α is true in w and $\neg\alpha$ is true in w'.

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[2] Another objection one might raise is that giving a heterodox interpretation to $\Diamond$ and at the same time keeping the same syntactic symbol might be, to say the least, confusing. While not disagreeing on that, we would just say that our decision to keep $\Diamond$ has to do with our purpose to emphasize that the plausibility component of our system is not a new logic whatsoever, but old traditional normal modal logic, and to make explicit, we might say, this so-called paraconsistent aspect of modal logic.

Here we arrive at an interesting parallel between our approach and the way default logic traditionally deals with contradictions. More specifically, considering that the consequent of defaults will always be of the form $\Diamond\alpha$, there will be a close relationship between our notion of plausible worlds and the extensions that would be generated by the corresponding $\Diamond$-less default rules. In traditional default logic, the contradictions that may eventually be inferred from a default theory T generate multiple self-consistent extensions, being then the parts of a contradiction accommodated in different extensions. Following the standard terminology, we might say that if a formula α belongs to all these extensions it is a *skeptical consequence* of T; if it belongs to at least one extension it is a *credulous consequence* of T.

Now, if we do like we are suggesting here and mark the consequent of defaults with $\Diamond$ and use some normal modal logic as our underlying monotonic logic, contradictory conclusions will be accommodated in the only set of conclusions, which of course will be satisfied by some Kripkean semantic model M. Given this model M, there will be a correspondence between the plausible worlds of M and the old extensions generated by traditional default logic: for each one of these extensions there will be one or more plausible worlds *w* of M satisfying it (provided of course *w* is interpreted as a non-modal valuation).

A consequence of this correspondence between extensions and plausible worlds on the one hand and our classification of skeptical and credulous consequences of a default theory above on the other is that it suggests a very interesting refinement of our notion of plausibility. Since there are two kinds of inductive consequences, a skeptical and a credulous one, we can say that there are two kinds of plausibility notions: a *skeptical* and a *credulous* one [9, 38]. And since each extension corresponds, grossly, to one plausible world, given the definition we gave above to the notions of skeptical and credulous inductive consequences, we have that while a credulously plausible statement is a statement true in at least one plausible world, a skeptically plausible one is a statement true in all plausible worlds. Thus, while the $\Diamond$ operator will trivially correspond to a *credulous notion of plausibility*, the notion of *skeptical plausibility* shall be perfectly captured by the operator $\Box$, which might be introduced as a derived symbol from $\neg$ and $\Diamond$ ($\Box\alpha =_{def} \neg\Diamond\neg\alpha$).

To finish this section, let us summarize in somewhat more precise terms how the two tasks of our project shall interact to build our so-called logic of induction. First, we shall consider an expansion of a specific modal calculus (interpreted as a calculus of plausibility) in such a way that inductive implications or defaults are added to the logical language and treated as atomic formulae by its axiomatic machinery. In this way we will be able to make defaults and ordinary formulae to interact with the help of standard logical connectives as well as to have defaults appearing as the prerequisite, justification or consequent of another default. Second, we shall have a mechanism capable of reasoning nonmonotonically on the inductive implications of our expanded calculus of plausibility, which in this case shall function as the monotonic basis of this nonmonotonic machinery.

4 A Logic of Inductive Implication

In this section we shall try to apply the ideas laid down the previous sections and build what we might call a calculus of inductive implication. As we have said, in order to do that we shall use as a starting point Reiter's default logic and a specific modal calculus interpreted as a calculus of plausibility. Our first definition constructs, from a (possibly modal) language $\Im$ containing the logical symbols $\neg$, $\rightarrow$, $\vee$, $\wedge$, $\forall$, $\bot$ and $\bot$ (with their usual interpretations), an inductive language $\Im_{\succsim}$ containing formulae of the form $\alpha \succsim \beta \not\precsim \varphi$:

Definition 1. Let $\Im$ be a language. The *inductive language* $\Im_{\succsim}$ built over $\Im$ is defined as follows:

- (i) If $\alpha \in \Im$ is such that it contains no one of $\Im$'s logical symbols, then $\alpha \in \Im_{\succsim}$;
- (ii) If $\oplus$ is a monadic logical symbol of $\Im$ along with its non-logical complements, if there is any, and $\alpha \in \Im_{\succsim}$, then $(\oplus\alpha) \in \Im_{\succsim}$;
- (iii) If $\oplus$ is a dyadic logical symbol of $\Im$ and $\alpha,\beta \in \Im_{\underline{\succ}}$, then $(\alpha \oplus \beta) \in \Im_{\succsim}$;
- (iv) If $\alpha,\beta,\varphi \in \Im_{\succsim}$, then $(\alpha \succsim \beta \not\precsim \varphi) \in \Im_{\succsim}$;
- (v) Nothing else belongs to $\Im_{\succsim}$.

Items (i)-(iii) just say that any formula belonging to $\Im$ also belongs to $\Im_{\succ}$; the real novelty is item (iv), which defines what we have called *inductive implications*. $\alpha \succ \beta \not\precsim \varphi$ means "α inductively implies β unless φ". We call any formula of $\Im_{\succ}$ that is not an inductive implication an *ordinary formula*. We call α the antecedent of $\alpha \succ \beta \not\precsim \varphi$, β its consequent and φ its exception. $\alpha \succ \beta$ is an abbreviation for $\alpha \succ \beta \not\precsim \bot$ and $\beta \not\precsim \varphi$ is an abbreviation for $\top \succ \beta \not\precsim \varphi$; while $\alpha \succ \beta$ means simply "α inductively implies β", $\beta \not\precsim \varphi$ can be read as "β is the case unless φ".

As we shall see in subsequent definitions, $\alpha \succ \beta \not\precsim \varphi$ intends to be equivalent to Reiter's default rule $\alpha{:}\beta \wedge \neg\varphi/\beta$. Here we can point to two significant differences between Reiter's approach and ours. First, instead of being rules of inference belonging to the meta-language, in our approach defaults belong to the logical language. Second, according to definition 1, inductive implications can freely interact with the other connectives as well as with other inductive implications. For example, if α, β, φ and λ belong to $\Im_{\succ}$, then the formulae below also belong to $\Im_{\succ}$:

(1) $\alpha \succ (\beta \succ \varphi \not\precsim \lambda) \not\precsim \phi$
(2) $(\beta \succ \varphi \not\precsim \lambda) \succ \alpha \not\precsim \phi$
(3) $\alpha \wedge (\varphi \succ \lambda \not\precsim \beta)$
(4) $(\alpha \succ \beta) \rightarrow ((\beta \succ \varphi) \rightarrow (\alpha \succ \varphi))$

Once we have an inductive language $\Im_{\succ}$ we can use a specific calculus based on language $\Im$ to very easily construct a calculus able to monotonically reason upon inductive implications. When this $\Im$-based calculus is a modal one functioning as a logic of plausibility (with $\diamond$, for example, being interpreted as "it is plausible that") we say that the extended, $\Im_{\succ}$-based calculus is a *pseudo-inductive logic of plausibility*. The term "pseudo-inductive" indicates that the calculus in question is deductive rather than inductive, but nevertheless contains and reasons (deductively) about inductive implications.

We shall represent a modal calculus M by a pair $<\Im, \Lambda>$, where $\Im$ is its (modal) language and Λ is its set of axiom schemas and inference rule schemas. The set of axioms of M shall be of course the set of all formulae of

$\Im$ satisfying at least one of the axiom schemas of Λ. We represent $\Im$'s set of modal operators by $\Theta(\Im)$. We also say that $\Im$ is based on $\Theta(\Im)$. Given this, we define the notion of pseudo-inductive logic of plausibility in two steps, defining first what we call a *pseudo-inductive modal logic*:

Definition 2. Let $M = <\Im,\Lambda>$ be a modal calculus. The *pseudo-inductive modal logic* M^* based on M is the modal calculus $<\Im_{\succ},\Lambda>$, where $\Im_{\succ}$ is the inductive language built over $\Im$. We shall use the symbol $\vdash_{M^*}$ to refer to M^*'s relation of inference.

Here the set of axioms of M^* is simply the set of all formulae of $\Im_{\succ}$ which satisfy at least one of Λ's axiom schemas, the same holding for its inference rules. In order to obtain a pseudo-inductive logic of plausibility from a pseudo-inductive modal logic we have just to choose one or more of its modal operators to play the role of our plausibility operator:

Definition 3. A *pseudo-inductive logic of plausibility* P is a pair $<M^*, \Theta^*>$ where $M^* = <\Im_{\succ},\Lambda>$ is a pseudo-inductive modal logic and $\Theta^* \subseteq \Theta(\Im_{\succ})$ is a set of modal operators.

As we have anticipated, the difference between P and M^* is that in P we have chosen a subset of $\Theta(\Im)$ to be the set of our plausibility modal operators. Because of that we can call it a *logic of plausibility*. With such a logic of plausibility we can, for example, use (4), which represents a sort of transitivity property of inductive implications, to conclude $\alpha \succ \varphi$ from $\alpha \succ \beta$ and $\beta \succ \varphi$, which is, we might say, a way to generate defaults from defaults. However, in order to really take advantage of inductive implications we must also be able to nonmonotonically infer the consequent of defaults, which shall allow us, for example, to represent ways of nonmonotonically generate inductive implications, such as is done in (1), for instance.

In order to do that, we shall use a construction very similar to default logic's: we shall define something akin to the notion of default theory, which we shall call a P-*theory* – P is a specific pseudo-inductive logic of plausibility – and then using a fixed point operator define what we shall call a P-*extension*. However, one must remember that in order to paraconsistently deal with inductive inconsistencies, we need that ordinary formulae appearing as consequents of inductive implications be marked with one of our plausibility operators. A consequence of that is that the set of formulae that might be used to compose a P-theory is a proper subset of $\Im_{\succ}$. In order to avoid additional complications, we shall also restrict ourselves to closed formulae. Below we have the definition of what we call a θ-*inductive formula*, which basically show how the members of Θ^* shall play the role of a plausibility modal operator:

Definition 4. Let $\Im$ be a modal language based on a set of modal operators Θ and $\theta \in \Theta$ a modal operator. The notion of θ-*inductive formula* is defined as follows:

- (i) If $\alpha \in \Im_{\succ}$ is of the form $\theta\varphi$, then α is a θ-inductive formula;
- (ii) If $\alpha \in \Im_{\succ}$ is a θ-inductive formula, then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ and $\forall x \alpha$ are also θ-inductive formulae;
- (iii) If $\beta \in \Im_{\succ}$ is a θ-inductive formula, then $\alpha \succ \beta \precnsim \varphi$ is a θ-inductive formula;
- (iv) Nothing else is a θ-inductive formula.

The idea of this definition is, given a distinguishing modal operator θ, to set which formulae of $\Im_{\succ}$ are admissible in a theory which intends to use $\alpha \succ \beta \precnsim \varphi$ as inductive implications and θ as a plausibility modal operator. Note that item (iii) requires that the consequent of an inductive implication be a θ-inductive formula, which in the case of ordinary formulae implies having modal formulae of the kind $\theta\beta$ as consequent of inductive implications. Supposing $\Im$ is the language of traditional modal logic and $\diamond$ is our distinguishing modal operator, we have below some examples of $\diamond$-inductive formulae:

(5) $\diamond(\alpha \rightarrow \neg\beta)$

(6) $(\alpha \succ \diamond\beta \precnsim \varphi)$

(7) $(\beta \succ \diamond\varphi \precnsim \lambda) \succ \diamond\alpha \precnsim \phi$

Given a pseudo-inductive logic of plausibility P and the notion of θ-inductive formula, we can define what we call P-*inductive language*, that is to say, the formulae of $\Im_{\succ}$ which can be used in the construction of a P-theory:

Definition 5. Let P = <M*, Θ*> be a pseudo-inductive logic of plausibility with M* = <$\Im_{\succ}$,Λ>. The P-*inductive language* $\Im_{P}$ is defined as follows:

(i) If $\alpha \in \Im_{\succ}$ is an ordinary closed formulae, then $\alpha \in \Im_{P}$;

(ii) If $\alpha \in \Im_{\succ}$ is a closed θ-inductive formula such that $\theta \in \Theta^*$, then $\alpha \in \Im_{P}$;

(iii) Nothing else belongs to $\Im_{P}$.

We call any set $A \subseteq \Im_{P}$ a P-*theory*.

Given then a pseudo-inductive logic P and a P-theory A, we can say what would be P-extension of A, that is to say, the monotonic and non-monotonic conclusions we can draw from A by interpreting formulas of the form $\alpha \succ \beta \precnsim \varphi$ as inductive implications as well as by making use of the deductive apparatus of P:

Definition 6. Let P = <M*, Θ*> be a pseudo-inductive logic of plausibility with M* = <$\Im_{\succ}$,Λ>, $A \subseteq \Im_{P}$ a P-theory and $S \subseteq \Im_{\succ}$ a set of closed formulae. $\Gamma(S) \subseteq \Im_{\succ}$ is the smallest set satisfying the following conditions:

(i) $A \subseteq \Gamma(S)$;

(ii) If $\Gamma(S) \vdash_{M^*} \alpha$ then $\alpha \in \Gamma(S)$;

(iii) If $\alpha \succ \beta \precnsim \varphi \in A$, $\alpha \in \Gamma(S)$ and $\neg\beta \notin S$ and $\varphi \notin S$, then $\beta \in \Gamma(S)$. A set of formulae E is a P-*extension* of A iff $\Gamma(E) = E$, that is, E is a fixed point of the operator Γ.

It is not hard to see the similarities between our definition of extension and Reiter's. We basically follow the same general idea of Reiter's definition of extension. First, item (i) guarantees that A belongs to its P-extension; second, due to item (ii), we have that every P-extension is deductively closed; third, item (iii) has the effect that as many inductive implications as possible will be used in the composition of the extension. Here, however, at item (iii), we find the first relevant dissimilarity. Following Buchsbaum and Pequeno [9], we make the test of consistency of the consequent inside the very definition of extension; this justifies not considering it among the exceptions of the inductive implication as well as our saying that $\alpha \succ \beta \precnsim \varphi$ means the same as $\alpha{:}\beta \wedge \neg\varphi/\beta$. And this, we must concede, is a quite natural and desirable thing. Among the exceptions to the claim that α inductively implies β one that will appear in all cases, independently of the form of α and β, is $\neg\beta$. Therefore, nothing more natural than not requiring $\neg\beta$ to be informed at every time we write an inductive implication. Proceeding in this way we do not, as we have mentioned, allow the representation of so-called abnormal defaults.

The main singularities of our approach, however, have to do with our use of a pseudo-logic of plausibility. First, instead of using classical logic as the underlying monotonic logic of our nonmonotonic machinery, we use an entirely different, however still monotonic, logic. So when we say that (ii) guarantees that a P-extension is deductively closed, we are of course meaning deductively closed *under* P. Second, we require ordinary formulae appearing as consequents of inductive implications to be marked with a plausibility modal operator, so as to prevent the appearance of formal contradictions and consequent splitting of extensions (in the case of course we follow the general idea presented here of using modal logic as a logic of plausibility). Third, we allow in the logical language the representation of defaults and chains of defaults, making us thus able to represent ways through which defaults are generated.

With the help of definition 6, we can define the notion of P-inductive consequence:

Definition 7. Let $P = \langle M^*, \Theta^* \rangle$ be a pseudo-inductive logic of plausibility with $M^* = \langle \mathfrak{I}_{\succ}, \Lambda \rangle$, $A \subseteq \mathfrak{I}_P$ a P-theory and $\alpha \in \mathfrak{I}_{\succ}$ a formulae. α is a P-*inductive consequence* of A (in symbols: $A \vdash_P \alpha$) iff, for all P-inductive extensions E of A, $\alpha \in E$[3].

3 From the point of view of the ordinary inductive consequences of a P-theory, there is no difference in using a credulous approach or a skeptical one (like we did above) in the definition of the relation of inductive consequence, for since our definition of P-theory forbids the appearance of formal contradictions, ordinary formulae will belong to the same extension. The cases where we might have more than one extension have to do with defaults

Let us now introduce the pseudo-logic of induction which we shall use in the rest of the paper. It is built upon modal logic S5 with $\Diamond$ as its primitive modal operator and meaning "it is plausible that." Despite the possibility and usefulness of using $\Box$ to represent a strong, skeptical notion of plausibility and have a pseudo-inductive logic of plausibility with two distinguishing modal operators, in our endeavor to exemplify our system we will restrict ourselves to the credulous plausibility represented by $\Diamond$.

Definition 8. Let S5* be the pseudo-inductive modal logic based on first order modal calculus S5. The pseudo-inductive logic of plausibility $P_{\Diamond}$ is the pair $<S5^*,\{\Diamond\}>$.

Given $A \subseteq \Im_{\gtrdot}$, where $\Im$ is the language of S5, the theory of A Th(A) is the set $\{\alpha \mid A \vdash_{S5^*} \alpha\}$.

Now, since a pseudo-inductive logic of plausibility does not set any property of inductive implications, it cannot perform the task of generating defaults we have agreed an inductive logic should perform. The only thing it does concerning formulae of the form $\alpha \succ \beta \precnsim \varphi$ is to detach the consequent from the antecedent. It is like a calculus of material implication provided with MP but with no axioms for $\rightarrow$. However, akin to such an implication-axiom-less calculus, a pseudo-inductive logic of plausibility provides the basic tools with which we can build so-called *inductive axioms* and obtain something worthy of being called a logic of induction or a calculus of inductive implication:

Definition 9. Let $P = <M', \Theta'>$ be a pseudo-inductive logic of plausibility with $M' = <\Im_{\gtrdot},\Lambda>$, $A \subseteq \Im_P$ a P-theory, $T \subseteq \Im_P$ a P-theory called the set of inductive axioms and $\alpha \in \Im_{\gtrdot}$ a formula. α is a T-P-*inductive consequence* of A (in symbols: $A \vdash_{T\text{-}P} \alpha$) iff $T \cup A \vdash_P \alpha$.

Definition 10. A *logic of induction* or *calculus of inductive implication* C is a triple $<P,T,\vdash_{T\text{-}P}>$ where $P = <M', \Theta'>$ is a pseudo-inductive logic of plausi-

with other defaults in their consequents. We postpone to the future an analysis of such bad-behaved P-theories.

bility with $M' = <\Im_{\succsim},\Lambda>$, $T \subseteq \Im_P$ is a P-theory representing the set of inductive axioms and $\vdash_{T\text{-}P}$ is the T-P relation of inductive consequence. We also refer to $\vdash_{T\text{-}P}$ as $\vdash_C$.

5 Hempel's Calculus of Confirmation and the Abduction and Hypothetico-Deductive Models

Now that we have introduced the basic conceptual aspects of our system, we may ask: Which sort of inductive implications are worth of being taken as inductive axioms? Considering that formulae of the form $\alpha \succsim \Diamond\beta \not\precsim \varphi$ are our representation of confirmation statements of the form "α inductively *confirms* β unless φ," it seems reasonable to try to answer this question by looking at some general conditions philosophers have proposed to characterize the minimal properties that every definition of confirmation is supposed to satisfy (which are more or less like the properties represented by a calculus of material implication which are supposed to set the basic properties of implication sentences.) The set of conditions we will examine here are the one proposed by Carl Hempel's in his classical paper of 1945 "Studies in the Logic of Confirmation" [19] added by a few more conditions proposed later by other philosophers.

In [19], Carl Hempel proposed a set of conditions which any model of confirmation is supposed to satisfy. In other words, independently of how one sets in which circumstances a piece of evidence confirms or inductively supports a hypothesis, the following restrictions should be satisfied:

- (I) *Entailment condition*: if statement α entails (i.e., logically implies) statement β, then β should be confirmed by α;
- (II) *Consequence condition*: if statement α confirms statement β and β logically implies statement φ, then α should also confirm φ;
- (III) *Equivalence condition*: if statement α confirms statement β and β is logically equivalent to φ, then α should also confirm φ;
- (IV) *Weak Consistency condition*: if statement α confirms statement β and α is not self-contradictory, then α and β should be logically compatible;

(IV') *Strong Consistency condition*: if statement α confirms statements β and φ and α is not self-contradictory, then β and φ should be logically compatible.

This list might be lengthened by some few additional conditions:

(V) *Inverse Equivalence condition*: if statement α confirms statement β and α is equivalent to statement φ, then φ should confirm β;

(VI) *Transitivity condition*: if statement α confirms statement β and β confirms statement φ, then α should confirm φ.

Together, the set of conditions I-VI entails the following derivate conditions:

(VII) *Inverse Consequence condition*: if statement α entails statement β and β confirms statement φ, then α confirms φ;

(VIII) *Inclusion Condition*: every statement confirms itself.

Formulations of V and VI have appeared, respectively, in [34] and [20]. VII is obtained from I and VI, and VIII is a special case of I. Since we shall interpret $\alpha \succsim \Diamond\beta \not\precsim \varphi$ as "α confirms $\Diamond\beta$ unless φ," condition IV' is automatically satisfied by $L_{\Diamond}$. What follows below are the axioms of what we shall call *Hempel calculus of confirmation*.

Definition 11. Let $P = \langle M', \Theta' \rangle$ be a pseudo-inductive logic of plausibility with $M' = \langle \Im_{\succsim}, \Lambda \rangle$. The *Hempel confirmation axioms* T_H in $\Im_P$ is the set composed by all formulae of $\Im_P$ satisfying the following schemas of formula:

I: $(\alpha\rightarrow\beta)\succsim(\alpha\succsim\Diamond\beta)\not\precsim((\alpha\leftrightarrow\bot)\vee(\top\leftrightarrow\beta))$ *Entailment*

II: $(\beta\rightarrow\varphi)\succsim((\alpha\succsim\Diamond\beta\not\precsim\phi)\rightarrow(\alpha\succsim\Diamond\varphi\not\precsim\phi))\not\precsim(\top\leftrightarrow\varphi)$ C*onsequence*

III: $(\alpha\succsim\Diamond\beta\not\precsim\phi)\rightarrow((\beta\leftrightarrow\varphi)\rightarrow(\alpha\succsim\Diamond\varphi\not\precsim\phi))$ *Equivalence*

IV: $(\alpha\succsim\Diamond\beta\not\precsim\varphi)\rightarrow((\alpha\wedge\beta\rightarrow\bot)\rightarrow\bot)$ *Weak consitency*

V: $(\alpha\succsim\Diamond\beta\not\precsim\phi)\rightarrow((\alpha\leftrightarrow\varphi)\rightarrow((\varphi\succsim\Diamond\beta\not\precsim\phi))$ *Inverse Equivalence*

VI: $(\alpha\succsim\Diamond\beta\not\precsim\phi)\rightarrow((\beta\succsim\Diamond\varphi\not\precsim\phi')\rightarrow(\alpha\succsim\Diamond\varphi\not\precsim\phi\vee\phi'))$ *Transitivity*

VII: $(\alpha\to\beta)\gtrsim((\beta\gtrsim\Diamond\varphi\lnsim\phi)\to(\alpha\gtrsim\Diamond\varphi\lnsim\phi))\lnsim(\alpha\leftrightarrow\bot)$ *Inverse Consequence*

VIII: $\alpha\gtrsim\Diamond\alpha$ *Inclusion*

Definition 12. Let T be Hempel confirmation axioms in $\mathfrak{I}_{P\Diamond}$, where $\mathfrak{I}_{P\Diamond}$ is pseudo-inductive logic of plausibility $P_\Diamond$'s language (that is to say, pseudo-inductive modal logic S5*'s language). The *Hempel calculus of confirmation* C_H is the triple $<P_\Diamond,T,\vdash_{T\text{-}P\Diamond}>$.

The reason for representing I, II and VII through an inductive implication instead of a material implication formula is due to inability of $\to$ to capture the relevance aspect required by a confirmation relation. If we represent I by $(\alpha\to\beta)\to(\alpha\gtrsim\beta)$, for example, we would have that for any sentence α and β, $\alpha\gtrsim\Diamond\top$ and $\bot\gtrsim\Diamond\beta$, i.e., the plausibility of a tautological formula is confirmed by α and a contradictory one confirms $\Diamond\beta$, for any two formulae α and β. And because we have represented I and II as inductive implications instead as material implication formulae, we cannot infer VII and VIII from them. Consequently we had to add them to our set of inductive axioms.

As it can be easily seen, the axioms of Hempel calculus of confirmation function like a calculus of confirmation, setting the general properties according to which confirmation statements are supposed to be obtained from prior, already existing confirmation statements. They however do not say anything about how these priori confirmation statements are supposed to be generated, which is the task of what we have been calling *model of confirmation*. We now examine how to obtain through our framework a representation of two of the most famous models of confirmation: the *hypothetico-deductive* (H-D) model and its brother-model the *abductive* model.

What we call the abductive model of confirmation is the confirmatory form of the so-called abductive reasoning: α confirms β if $\beta\to\alpha$ (if α is true we then get that β is plausible.) It is present in Hempel's 1945 article in the form of what he calls the *Converse Consequence condition* (condition IX in his numbering): if statement α confirms statement β and statement φ logically implies β, then α confirms φ. This along with VIII implies the *Converse Entailment condition* (item X): if statement α logically implies statement β, then β confirms α. Now we may think that all we have to do to obtain a Hempelian abductive model is to add these conditions to Hempel calculus of confirmation. Not quite so. As Hempel showed, these two conditions are incompatible with his previous conditions.

Consider for example the Stark effect (Se) which is known to confirm quantum mechanics (Qm). Let also Bl be the hypothesis that black cats bring bad luck. Trivially, $Qm \wedge Bl \rightarrow Qm$. But since Se confirms Qm, we have by IX that Se confirms $Qm \wedge Bl$. Now, given that $Qm \wedge Bl \rightarrow Bl$, by II we have the absurd conclusion that Se supports the hypothesis that black cats bring bad luck. In fact, Se will confirm not only Bl, but any statement expressible in the language at hand. Things get still worse when we consider X, which leads to the conclusion that any pair of statement *e-h* is such that *e* confirms *h*. The same unwanted conclusion could be derived if we consider disjunctive statements rather than conjunctive ones. Since $\alpha \rightarrow \alpha \vee \beta$, by I we have that α confirms $\alpha \vee \beta$. But since $\beta \rightarrow \alpha \vee \beta$, by IX we have that α confirms β. Similarly, by X we have that $\alpha \vee \beta$ confirms β. Since by I α confirms $\alpha \vee \beta$, by the transitivity condition we have that α confirms β.

Because of these problems, Hempel rejected X along with the definition of confirmation which brought it into the discussion: the *prediction-criterion of confirmation*. This prediction-criterion of confirmation is nothing less than a formulation of the so-called *hypothetico-deductive model* of confirmation, whose importance for the contemporary theory of science is such that some philosophers went so far as claiming it to be the official "scientist's philosophy of science" [26]. Hempel's formulation of the H-D model goes as follows:

α confirms β if the three conditions below are satisfied:

(i) $\vdash \alpha \leftrightarrow \alpha \wedge \alpha$";

(ii) $\{\alpha' \wedge \beta\} \vdash \alpha$";

(iii) $\{\alpha'\} \nvdash \alpha$".

In words: if α is composed by two statements α' and α" and α" can be logically deducted from α' in conjunction with β but not from α' alone, then β is confirmed by α. Since this definition of confirmation satisfies conditions IX and X, it is easy to conclude that it will be plagued by the same problems we have shown in connection with the abductive model [17].

About the cause of these problems, in contrast to what many philosophers have held, it is not on the H-D and abductive models themselves, but in the tools we are using to represent them. In a nutshell, they all come from the irrelevance feature of classical entailment [39]. When, for instance, we say that if α entails β then β confirms α, we expect that all parts of α are necessary for the derivation of β and therefore deductively connected with it. Now, if α and β are so connected and we conjoin φ with α, trivially β is logically entailed by $\varphi\wedge\alpha$; but φ plays no role at all in the derivation of β from $\alpha\wedge\varphi$. Therefore we are not in any way ready to say that $\alpha\wedge\varphi$ confirms β, even though α alone does. The same thing happens when we take the disjunction of β and φ. All the incompatibility between IX and X and Hempel's former conditions as well as the problems we have identified with axioms I and II come from this irrelevance feature of classical entailment.

What follows now is an attempt to formulate the abductive and H-D models inside our framework which is not plagued by the mentioned problems. We first define a few abbreviations:

(i) $\alpha\Leftrightarrow\beta =_{def} (\alpha\rightarrow\beta)\vee(\beta\rightarrow\alpha)$;

(ii) $\alpha\rhd\beta =_{def} \beta\succ\Diamond\alpha$;

(iii) $\alpha\trianglerighteq\beta =_{def} (\alpha\rightarrow\beta)\wedge(\alpha\rhd\beta)$;

(iv) $\alpha\ntriangleright\beta =_{def} (\alpha\rhd\beta)\succ\Diamond\bot$.

$\alpha\Leftrightarrow\beta$ is a simple abbreviation meaning that α and β are "implicationally connected" to each other[4]. $\alpha\rhd\beta$ is an alternative way of writing $\alpha\succ\Diamond\beta$ which will be of some help in our task of representing the abductive method of confirmation. $\alpha\rhd\beta$ can be read as "α is confirmed by β." $\alpha\trianglerighteq\beta$ is intent to represent a situation where α relevantly implies β. It depends directly on what we have called abduction model of confirmation: if α (relevantly) implies β, then β confirms α. That is to say, supposing that we have such a model, if β confirms α and $\alpha\rightarrow\beta$, then α relevantly implies β. Finally, $\alpha\ntriangleright\beta$ represents a situation where α and β are such that, due to the lack of a rele-

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[4] For clarity purposes we found it worthy to include this abbreviation, even knowing that $(\alpha\rightarrow\beta)\vee(\beta\rightarrow\alpha)$ is a theorem in classical logic and consequently in S5.

vant entailment connection between α and β, α cannot be confirmed by β through the abduction model. We show below the basic axioms which will make use of these abbreviations.

Definition 13. Let P be a pseudo-inductive logic of plausibility. The *Abduction axioms* T_{Ab} in $\mathfrak{I}_P$ is the set composed by all formulae of $\mathfrak{I}_P$ satisfying the following schemas of formula:

X: $(\alpha\rightarrow\beta)\succ(\alpha\triangleright\beta)\precnsim(\alpha\ntriangleright\beta)$

Ab1: $((\alpha\leftrightarrow\alpha'\wedge\alpha'')\wedge(\alpha'\trianglerighteq\beta))\succ(\alpha\ntriangleright\beta)\precnsim((\alpha''\trianglerighteq\beta)\vee(\alpha'\Leftrightarrow\alpha''))$

Ab2: $((\beta\leftrightarrow\beta'\vee\beta'')\wedge(\alpha\trianglerighteq\beta'))\succ(\alpha\ntriangleright\beta)\precnsim((\alpha\trianglerighteq\beta'')\vee(\beta''\Leftrightarrow\beta'))$

The purpose of the above axioms is basically to define what we have been calling abductive confirmation. X, which is a more sophisticated formulation of the converse entailment condition, sets the basic abductive criterion according to which formula α confirms formula β: if $\alpha\rightarrow\beta$ then α is confirmed by β. However, as we have seen, material implication does not embody the relevant aspects required by an abductive model of confirmation: sometimes even though $\alpha\rightarrow\beta$, due to α's not being relevantly connected with β, β is not confirmed by α. It is the goal of the exception part of X, $\alpha\ntriangleright\beta$, to block these non-relevance cases and therefore prevent $\alpha\triangleright\beta$ from being concluded from $\alpha\rightarrow\beta$. These non-relevance cases are formally defined by axioms Ab1 and Ab2, which basically take into account the conjunction and disjunctive problems of abduction which we have discussed above.

Ab1 says that if α is equivalent to the conjunction of α' and α'', and α' relevantly implies β, then to conjoin α' and α'' and write $\alpha'\wedge\alpha''\rightarrow\beta$ will be a trivialization with no relevance content. Therefore $\alpha\ntriangleright\beta$. Of course there are exceptions to this. The first one is α'' relevantly implying β, in which case $\alpha'\wedge\alpha''$ should be confirmed by β (which will be obtained by using $\alpha'\wedge\alpha''\rightarrow\beta$ along with X.) Also, if $\alpha'\rightarrow\alpha''$ or $\alpha''\rightarrow\alpha'$ then $\alpha\ntriangleright\beta$ should not be the case, for if $\alpha'\rightarrow\alpha''$ then α' will be equivalent to $\alpha'\wedge\alpha''$, and if $\alpha''\rightarrow\alpha'$, by transitivity $\alpha''\rightarrow\beta$ and therefore $\alpha''\triangleright\beta$. Hence, $\alpha'\wedge\alpha''\triangleright\beta$. One could think that this second part of $\alpha'\Leftrightarrow\alpha''$ was not needed at all, for since $\alpha''\triangleright\beta$ (which is obtained by using X along with $\alpha''\rightarrow\beta$) the situation was already contemplated by Ab1. However, taking Ab1 without $\alpha''\rightarrow\alpha'$ in its exception part and $\alpha''\rightarrow\alpha'$ and $\alpha'\trianglerighteq\beta$ as valid formulae (which implies $\alpha''\leftrightarrow\alpha'\wedge\alpha''$) entails a conflict between X and Ab1: by using Ab1 first and concluding $\alpha''\ntriangleright\beta$ (which could be done because we have not used yet X to conclude $\alpha''\triangleright\beta$ and be able to block Ab1) we will not be able to use X and conclude $\alpha''\triangleright\beta$. Therefore two extensions would arise. In order to prevent that, we have to consider $\alpha''\rightarrow\alpha'$ in the very exception part of Ab1.

For Ab2 the reasoning is almost the same. If β is equivalent to the disjunction of β' and β'', and α relevantly implies β', then to write $\alpha\rightarrow\beta'\vee\beta''$ means to go against our relevance principle, for β'' plays no role at all in the derivation of $\beta'\vee\beta''$ from α. Therefore $\alpha\ntriangleright\beta$. About the exceptions, we have first that if α relevantly implies β'' then α should be confirmed by $\beta'\vee\beta''$. Also, if $\beta''\rightarrow\beta'$ or $\beta'\rightarrow\beta''$ then $\alpha\ntriangleright\beta$ should not be the case, for if $\beta''\rightarrow\beta'$ then β' will be equivalent to $\beta'\vee\beta''$, and if $\beta'\rightarrow\beta''$ then by transitivity $\alpha\rightarrow\beta''$ and therefore $\alpha\triangleright\beta''$. Hence $\alpha\triangleright\beta'\vee\beta''$. About the objection that it is not necessary to consider $\beta'\rightarrow\beta''$ as an exception, taking Ab2 without $\beta'\rightarrow\beta''$ in its exception, and $\beta'\rightarrow\beta''$ and $\alpha\trianglerighteq\beta'$ as valid formulae (which implies $\beta''\leftrightarrow\beta'\vee\beta''$) entails a conflict between X and Ab2: by using Ab2 first and concluding $\alpha\ntriangleright\beta''$ (which could be done because we have not used yet X to conclude $\alpha\triangleright\beta''$ and be able to block Ab2) we will not be able to use X and conclude $\alpha\triangleright\beta''$. Therefore two extensions would arise.

Below we have what we can call the abduction logic of induction.

Definition 14. Let T be the abduction axioms in $\mathfrak{I}_{P\diamond}$. The *abduction model of confirmation* C_{Ab} is the triple $<P_{\diamond},T,\vdash_{T\text{-}P\diamond}>$.

With these abduction axioms at hand we can also define our version of the H-D model.

Definition 15. Let P be a pseudo-inductive logic of plausibility. The *H-D axioms* T_{Ab} in $\mathfrak{I}_P$ is the set composed by all formulae of $\mathfrak{I}_P$ satisfying the following schema of formula:

$$\text{H-D: } (\beta \leftrightarrow \beta' \wedge \beta'') \wedge (\alpha \wedge \beta' \rightarrow \beta'') \succ (\beta \succ \Diamond \alpha) \not\precsim (\alpha \wedge \beta' \not\rhd \beta'') \vee (\top \rightarrow \alpha)$$

Here we are using the formulation proposed by Hempel which we have shown at the beginning of this section. There will be two kinds of exceptions to this rule. The first obviously are situations where $\alpha \wedge \beta'$ does not relevantly imply β''. The second are cases where α is a tautology. The reason for this second sort of exception is that since $\beta' \leftrightarrow \top \wedge \beta'$, we do not want to take $\top \wedge \beta' \rightarrow \beta''$ as an irrelevant implication. Therefore $\top \wedge \beta' \not\rhd \beta''$ will not be the case. But we are also not ready to say that $\beta' \wedge \beta''$ confirm $\top$. The only alternative then is to consider this case as a separated exception. Concerning Hempel's three conditions, we note that the third one ($\{e'\} \nvdash e''$, which would be represented in our notation by introducing $\beta' \rightarrow \beta''$ in the exception part of H-D) is already contemplated by $\alpha \wedge \beta' \not\rhd \beta''$.

We then finally define what we call H-D model of confirmation:

Definition 16. Let T_{Ab} and $T_{H\text{-}D}$ be the abduction axioms in $\Im_{P\Diamond}$ and the H-D axioms in $\Im_{P\Diamond}$, respectively. The *H-D model of confirmation* $C_{H\text{-}D}$ is the triple $<P_{\Diamond}, T_{Ab} \cup T_{H\text{-}D}, \vdash_{T\text{-}P\Diamond}>$.

6 Conclusion

In this work we have tried to establish some connections between the field of nonmonotonic logic and the philosophical field of inductive logic. We have tried to materialize our conclusions by proposing a logical system inspired in Reiter's default logic that minimally fulfills the purpose of a logic of induction. Among the distinguishing features of his system, we have that it is able to represent chains of inductive rules as well as to reason paraconsistently on the conclusions obtained from them. In order to show the usefulness of our system, we tried to show how it can be used to repressent some traditional ideas concerning confirmation theory, more specifically the ones contained in Carl Hempel's paper "Studies in the Logic of Confirmation" and the ones incorporated in the so-called abductive and hypothetico-deductive models. Even though our formalization of these ideas was perhaps oversimplified and, some could say, somehow *adhoc* (we are thinking about the relevant aspect of the H-D model), we think it was useful in showing the fruitfulness of our approach.

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