CONDITIONAL ANTINOMIES

Claudio Pizzi *

Abstract: After a short premise about some paradoxical features of counterfactual reasoning, the paper tries to identify the analogues of the so-called Card Paradox and of the Liar antinomy by using a language containing the conditional operator symbolized by >. A first proposal is taken into account but dismissed since the resulting Liar is equivalent to the statement "the present statement is necessarily false", which makes the corner operator indistinguishable from strict implication. A second proposed solution consists in paraphrasing every statement *A* into the conditional "I would lie if I said not-A", where the conditional has the properties of the classical conditional operator. The "Epimenides" and "Truth-Teller" variants of the paradox are also examined in the last section.

Keywords: Antinomy. Paradox. Liar. Epimenides. Counterfactuals. Conditionals.

*Emeritus Siena University. Born in Milan (Italy) in 1944. Main research interests: Tense Logic, Conditional Logic, Philosophy of Causality, Abductive Reasoning.

§1. Conditionals have been a source of paradoxes since the beginning of contemporary logic, as it is shown by the so-called paradoxes of material and strict implication. However, their more paradoxical features came out especially after the II World War, along with the first attempts to determine the logical properties of counterfactual conditionals[1]. It is in itself paradoxical, to begin with, that a contrary-to-fact antecedent has normally two legitimate consequents which are incompatible. For instance, under standard presuppositions about Apollo and human kind, both the following counterfactuals appear to be justified:

(i) If Apollo were a man he would be mortal

(ii) If Apollo were a man he would be the instance of an immortal man

The paradox is here solved by the fact that one of the two conditionals, i.e. (ii), is "less intuitive" than the other. This asymmetry may be explained in terms of the possible world semantics which David Lewis associated to conditionals: the worlds in which Apollo is a man and is mortal are more similar to the actual world than the worlds in which Apollo is a man and is immortal. This explains why (ii) should be rejected and (i) should be accepted.

A different approach to conditionals is the Chishom-Goodman-Reichenbach one, according to which the truth of a conditional depends on a consequential nexus between the clauses. We may propose a definition of a consequential nexus in the following way.

We observe that (i) turns out to be more acceptable than (ii) since the consequent of (ii) implies the negation of a law of nature, i.e the removal of something which has a high information content, while in (i) the consequent implies the removal of a singular fact, whose information content is lower that the one of any law of nature. The latter consideration allows defining the consequential nexus between A and B, in a conditional of form $A > B$, as something which subsists when $A > B$ is more acceptable, in the defined sense, than $A > \neg B$. On the ground of such definition of truth, the truth of $A > B$ implies that $A > \neg B$ is false, or alternatively that $\neg(A > \neg B)$ is true. The rule $A > B \vdash \neg(A > \neg B)$ is sometimes called "Boethius' rule".

The main problem yielded by both the consequentialist and the Lewis' semantics for conditionals is that there are cases in which two incompatible consequents appear to be equally acceptable, so that we lack a criterion to choose one of the two rival conditionals. On the pattern of the famous Bizet-Verdi case proposed by Quine we can build an unlimited number

[1] For a survey of the logic of conditionals from both a historical and theoretical viewpoint see (ARLO-COSTA, 2007).

of examples. For instance, just to remain in the field of Italian opera, one could agree on the truth of the following two statements and on the fact that they transmit an equal amount of information:

(1) Verdi wrote the air “La donna è mobile”

(2) I hate the air “La donna è mobile”

So the following two conditionals appear to be equally acceptable, the first in view of (1) and second in view of (2):

(a) If I had been Verdi I would have written “La donna è mobile”

(b) If I had been Verdi I would not have written “La donna è mobile”

May we qualify as antinomies the paradoxes of the Bizet-Verdi family? An antinomy is by definition a contradiction derived from consistent premises. The notion of derivation which is used here is the one defined in standard first order logic, which offers the framework to reconstruct not only the Liar antinomy but all the well-known antinomies developed during the so-called “Crisis in the Foundations of Mathematics” (Cantor, Burali-Forti, Berry, Russell ...). But the implication relation occurring in (a) and (b) and usually represented by the conditional operator symbolized by “>” does not refer to the logical relation of derivation as above defined. The corner operator is not even reducible to strict implication, i.e. to necessary implication. David Lewis characterized counterfactual conditionals as *variably strict* conditionals, this meaning that in $A > B$ the consequent B follows not from A but from A conjoined with the “quasi-variable” $w(A)$, where $w(A)$ represents the conjunction of the most informative elements of the Background Knowledge compatible with A. If “$\supset$”stands for the material conditional and “$\Box$” for the necessity operator as axiomatized in KT, then a definition of the corner operator might be given as follows:

(Def >) $A > B =_{df} \Box ((w(A) \wedge A) \supset B)$

The minimal requirement we may put on $w(A)$is provided by the axiom

$(\Diamond\, w)\ \Diamond A \supset \Diamond\ (w(A) \wedge A)$

So the paradoxes of the Bizet-Verdi family are not properly speaking antinomies. The same could be said of other counterfactual paradoxes known in the literature. For instance, the following one, which stems from a puzzling question which was loved by Abraham Lincoln:

(3) If the tail of a dog were called "paw", how many paws would have a dog?

The antecedent of counterfactual question conjectures the existence of a language which is different form the one employed in the very formulation of the question itself. This class of counterfactuals could be called class of *Counterlinguistic Conditionals*.
Two incompatible answers to the question (3) seem to be equally plausible:

(4a) If the tail of a dog were called "paw", the dog would have five paws

(4b) Even if the tail of a dog were called "paw", the dog would have four paws.

Other paradoxes of the counterfactuals stem from a supposed illegitimacy of the antecedent. This problem is especially serious for consequentialists inasmuch the derivation of the consequent, from their viewpoint, is essentially granted by a certain set of natural laws L_1, L_2.... Let us suppose that the conjunction of laws is a certain theory CL. What happens if we suppose just the falsity of CL or of some of its components? We are in front of the conditionals which Goodman calls $\dashv$ conditionals. The answers given to the puzzle have been various. In the first place, we are allowed to draw conclusions from CL provided that in the derivation no use is made of laws depending on CL. But if we need such laws to draw a counterfactual conclusion we have to do with a self-defeating argument. In this case one may choose, for instance, to consider all counterlegals false or to deny that they have a truth-value[2].

The problem of counterlegal antecedents has actually some kinship with the Liar antinomy, but is not stricly speaking an antinomy. We may even make the absurd supposition that all the known laws of nature are false, but on this premise one could anyway draw consistent

[2] The former solution is implicit in Goodman's theory, while the latter has been proposed by Reichenbach

conclusions by using simply the logical laws of the background logical system. Making a further step, supposing the falsity of some law of logic L_i (e.g. the Excluded Middle) means to hypothesize a contradiction only if the background logical system S contains L_i , but not in any weaker system S' not including L_i.

§2. The question we now want to discuss is whether a Liar-like antinomy exists which can be written in a conditional language. As is well known, the Liar should not be confused with the Epimenides Paradox or its variants which belong to the class of what is called the family of Prior's Paradoxes[3]. Furthermore, the Liar has many variants, some of which may be introduced as generalizations of it. An example is the so-called "Card Paradox", originally due to the English logician Philip Jourdain, which can be simplified as follows:

$\mathbf{a}_1$: The statement written in the next line is true

$\mathbf{a}_2$: The statement written in the preceding line is false It is pointless to consider here that the two-vertices graph containing a 1 and a 2 may be generalized to a closed figure having an even number of vertices, connected by arrows, in which there is an alternation of statements of the following kind:

$\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$ ... The statement written at the next vertex is true

$\mathbf{a}_2$, $\mathbf{a}_4$, $\mathbf{a}_6$ ... The statement written at the next vertex is false

It is easy to show, for every closed figure of a finite length having the given pattern, that the statements in the graph yield an antinomy[4]. In case of a figure in which the vertices a_n and a_{n+1} are identical, i.e. $a_n = a_{n+1}$, there is not one but two possible self-referential statements: the standard Liar ("this statement is false") and the standard Truth teller ("this statements is true").

Another variant of the Liar is "Pinocchio's Paradox", which relies on the acquired knowledge of the fact that Pinocchio's nose grows if and only if he is lying. The paradox is provided

[3] See (PRIOR, 1961).
[4] It is to be noted that an antinomy follows even if the predicates "true"' and "false"' in the definition of the figure are inverted. An antinomy also follows from simply stipulating that the truth value of the even vertices is different from the truth value of the odd vertices.

by the statement

(5) Pinocchio says: “my nose is growing now”.

Is Pinocchio lying or not[5]?

Let us begin by trying to define a conditional version of the Card Paradox.

The first step is to imagine that the two sides of the card contain conditional statements, not categorical statements. In order to simplify the exposition we will assume that the background system is a minimal system of classical conditional logic. We choose the system named by Chellas CK [6] extended with ID +MP, i.e. the two axioms

(ID): $A > A$

(CMP): $(A > B) \supset (A \supset B)$

We will also assume a definition of the necessity operator in conditional terms

(Def $\Box$)$\Box A =_{df} \neg A > A$

which is of course equivalent to $\Box A =_{df} \neg A > \bot$

As a consequence of the Conditional Modus Ponens (CMP), the behaviour of $\Box$ includes the law T: $\Box A \supset A$. We will then also assume that the logic of $\Box$ is provided by the normal system KT.

Let us now define two statements X and Y in the following way:

[5] See (ELDRIDGE-SMITH, 2010).

[6] The axioms of CK, subjoined to the standard PC-calculus, are $(A > (B \wedge B')) \supset (A > B \wedge A > B')$, $(A > B \ \wedge \ A > B') \supset A > (B \wedge B')), A > \bot$. The rules are $A \equiv A'/A > B \supset A' > B$ and $B_1 \wedge B_2 \wedge ...B_n/A > B_1,...,A > B_n$. See (CHELLAS, 1975).

X: X is true > Y is true

Y: Y is true > X is false

It is inessential (but we can do it) to suppose the existence of a couple of points a_1 and a_2 (the card) such that a_1 is associated to X and a_2 is associated to Y. > is not a transitive relation, so we are not allowed to derive by transitivity the conclusion that X is true > X is false. Due to CMP, however, an antinomy is easily derived. In fact from X and Y we have straightforwardly

X': X is true ⊃ Y is true

Y': Y is true ⊃ X is false.

Now if X is true, by CMP X' is true. But if X' is true, given that X is by hypothesis true, Y is true by standard Modus Ponens. Thus Y' is also true, so by the antecedent of Y' and standard Modus Ponens it follows that X is false: contradiction. Since the supposition that X is true leads to a contradiction, this amounts to a proof of $\neg X$.

(b) Suppose X is false. Here we face the problem of establishing what we intend by negating a conditional. In order to avoid the introduction of a possible world semantics, we may simply look at the definition of > in terms of quasi-variables proposed at p. 1. According to such a definition the negation of X, i.e. $\neg(X$ is true $> Y$ is true), amounts to $\Diamond((w(X) \wedge$ X) true and Y false). So we have

(6) $\neg X \vdash \Diamond(w(X) \wedge X \wedge \neg Y)$

and by KT

(7) $\Diamond(w(X) \wedge X \wedge \neg Y) \vdash \Diamond X$

So by (6) and (7)

(8) $\neg X \vdash \Diamond X$

Now we showed before that X is provably false, so $\neg X$ is provably true. So by Necessitation $\Box\neg X$ is also provable. Then by (8) and Adjunction $\neg X \vdash \Diamond X \wedge \Box\neg X$: contradiction.

If we pass from the Card Paradox to the standard Liar we have to do with the statement which says of itself that, if it is true, it is false:

X: X is true > X is false

The conclusion to be drawn here is especially simple since $X > \neg X$ is by definition equivalent to $\Box \neg X$. X is then equivalent to stating that X is necessarily false. So if X is true, X is necessarily false, which yields a contradiction by the instance of T $\Box\neg X \supset \neg X$. So there is a proof that X is false, i.e. a proof of $\neg X$. If there is a proof that X is false then $\Box\neg X$ is a theorem by Necessitation. But $\Box\neg X$ is $X > \neg X$, so X: contradiction.[7]

A variant of X which should be taken into account is what one obtains in exchanging true and false in X:

X-: X- is false > X- is true

According to the definition of $\Box$, X- amounts to saying of itself that is necessarily true, which obviously does not lead to any contradiction. We could consider this assertion a conditional variant of the Truth Teller. Suppose in fact that it is false: then $\Diamond(w(\neg X) \wedge \neg X)$ and $\Diamond\neg X$, which is not only consistent with X- but in KT logically follows from the supposition $\neg X$.

§3. The systems of classical conditional logic axiomatize an operator > which has an intermediate strength between the material conditional and the strict conditional -3. As already remarked, we will have among the theses the wff $\bot > A$. But according to the consequentialist viewpoint which has been outlined at the beginning, this formula cannot be logically valid. From a consequentialist viewpoint, in fact, one should accept what is sometimes called Strawson's thesis or (weak) Boethius' Thesis:

(BT) $A > B \supset \neg(A > \neg B)$

(BT) states that $A > B$ e $A > \neg B$) cannot be both true. This fact is conflicting with the thesis $\bot > A$, since $\bot > A$ and $\bot > \neg A$ are both valid.

From (BT), via the identity law $A > A$, one obtains by Uniform Substitution and Modus Ponens the so-called Aristotle's Thesis

[7] The proof has the same structure of the one provided for the so-called Knower Paradox in (PRIEST, 1991, p.196)

(AT) $\neg(A > \neg A)$.

In any system with Uniform Substitution and the law $\neg\neg AA$, (AT) is equivalent to the following:

(AT-) $\neg(\neg A > A)$

From both AT and AT- it follows that $A > \neg A$ and $\neg A > A$ are logical contradictions and that they are both equivalent to $\perp$ so provably PC- equivalent.

If > has such properties, the conditional Liar turns out to be especially simple. In this new interpretation of the corner operator the statement

X*: X* is true > X* is false

turns out to be equivalent to

X*- : X* is false > X* is true

So the distinction between the Liar in the two forms (X) and (X-)(see page 6) here disappears. What it turns out is that both X* and X*- are simply contradictions, being both equivalent to $\perp$.

We should open a reflection on the problem whether that **X*** or **X*-** are actually antinomies. As a matter of fact, they have the logical form of a wff which is the negation of a thesis w.r.t. the background system, i.e. of a contradiction. This cannot be said of $X : X > \neg X$. If the background system is CK, X is equivalent to $\Box\neg X$, which lacks the logical form of a contradiction.

§4. An objection to the treatment of the conditional antinomies outlined in §3 is that it can be reproduced for any conditional operator satisfying the minimal conditions required for >. It is easy to show, for instance, that the axioms of CK+ID+MP are theorems if the conditional operator is replaced by the operator of strict implication. A further step consists in recalling that the box operator is defined in systems of strict implication as

(Def □) $\Box A =_{df} \neg A \dashv A$

which allows concluding that $\neg A \dashv A$ in classical conditional systems turns out to be equivalent to $\neg A > A$. So the proposed conditional Liar (X) is equivalent to the same statement in which $>$ is replaced by $\dashv$.

A point of discrimination between $>$ and $\dashv$ would be made by introducing for $>$ such strong axioms as $(A \wedge B) \supset \neg A > B$ or the so-called Conditional Excluded Middle $A > B \vee A > \neg B$: both principles do not hold for strict implication, but they are irrelevant to build the above reported antinomic argument.

We recall that the conceptual distinction between conditional statements and statements of strict implication is that the truth of the former (especially when they are counterfactual) may depend on a set of presuppositions compatible with the antecedent, which, as we saw, is here formally represented by the quasi-variables **w**(A), **w**(B), **w**(C) ... conjoined to the antecedents of the conditional. Due to the occurrence of such contextual elements of information, the conditional operator $>$ lacks some of the properties of strict implication, and especially Monotonicity, Contraposition and Transitivity.

A problem which deserves attention now is to find an antinomic argument which makes use of the $>$ - operator which is unequivocally endowed with the properties of a context-dependent operator.

To begin with, we could see whether there are ways to convert categorical statements into statements with the same meaning involving context-dependent conditionals. As a matter of fact, there is a variety of ways to perform such a translation. Let us take this example:

(9) My cat is black

We list here four ways to paraphrase this statement into one involving context- dependent conditionals:

i) Anyone who were to see my cat would see that it is black

ii) Any object that were identical to my cat is black

iii) If there is no error in what I am saying, my cat is back

iv) I would be a liar if I said that my cat is not black.

On this set of proposed translation the remarks are as follows:

a) The paraphrase in i) is inspired to the so-called phenomenalism as proposed originally by Stuart Mill. It applies only to statements which are reports of observation and not to other kinds of statements, e.g. to logical or analytical propositions. However, it could be modified in various ways, e.g. by saying "Anyone who were to know all the properties of my cat would know that it is black". But only an omniscient ideal subject may know *all* the properties of my cat, so that the antecedent supposes an impossibility and may be qualified as a particular counterlegal supposition (see p. 2).

b) The conditionals occurring in paraphrase of class 2) are called by Goodman *counteridenticals*. The paraphrase could be applied also to non-observational statements by use of propositional quantification. E.g. $\forall p$ (if p were a proposition identical to 2+2=4 it would be true). However, here the operator > has the same sense of necessary implication, so we are back to the trivialization deplored in the preceding sections.

c) The statement 3) has not exactly the sense of "my cat is black". Its sense is that the truth of what is asserted depends on the supposition that there are no mistakes in what I am saying. Its sense is more plausibly rendered by "it is likely that my cat is black".

d) The paraphrase 4) appears to be the more proper rendering of (9). To argue for this conclusion, we simply reason as follows. If it is true that my cat is black, then I would lie if I were to deny this fact, so the statement in 4) is true. If it is false, I would say something true in asserting that my cat is black, so I would say something false in denying it.

The formal rendering of the paraphrase in 4) is not simple. We will follow Prior (1961) in introducing a variable d for monadic functors appliable to propositions, of the kind which he uses in his reconstruction of the Epimenides argument. Here we will use the same symbol, with the meaning of "a says that -"[8] . This operator is simply governed by the axiom

(Axd) $dA \supset \neg d \neg A$

[8] The term "a" stand for any proper name of a human being, including the speaker (so "I" could also replace a).

We can also introduce two constant **F** and **T** with the meaning of False and True, such that d**F** means "a says a Falsity", "a says the Truth". A mixed axiom involving d, **F** and **T** is the following:

(LEM) $\neg(dA > d\mathbf{F}) \equiv d(A) > d\mathbf{T}$

We may write L (for "a is lying" or "a is a liar") in place of d**F** and ¬**L** in place of d**T**.

Let us now introduce an axiom which expresses the translation of any statement A in a context-depending conditional having the same meaning of A:

(Ax ≡) $A \equiv d(\neg A) > \mathbf{L}$

By uniform substitution in Ax≡ then:

(Ax≡') $\neg A \equiv dA > \mathbf{L}$

A noteworthy property of (Ax≡) is that it would be untenable if > were replaced by ⥽. In fact in this case we could perform the following proof:

(10) $A \supset (d\neg A \multimap \mathbf{L})$

then by Monotonicity of ⥽

(11) $A \supset ((d\neg A \wedge \neg A) \multimap \mathbf{L})$

But if A is true (11) is a false statement. In fact $(d\neg A \wedge \neg A) \multimap \mathbf{L}$ means that if A is false and I am saying that A is false this means that I am lying. This is clearly wrong since $d\neg A \wedge \neg A$ implies that I tell the truth by saying $\neg A$. So A implies a falsity. The relevant point is that the step from (10) to (11) cannot be done with > in place of ⥽, since > is a non-monotonic operator and does not allow so-called Weakening[9]

[9] In some systems of classical conditional logic Weakening is allowed with some restriction, for instance in the weak form $\neg(A > \neg B) \supset (A > C \supset ((A \wedge B) > C)$. This theorem is however not at disposal in CK + Id +MP.

Let us now examine the self-referential statement X whose intuitive meaning is "If a were to say the present conditional a would lie"

(12) $X : dX >\mathbf{L}$

Now substituting X to A in the equivalence Ax≡' one reaches the following equivalence

(13) $\neg X : dX >\mathbf{L}$

But according to (Ax ≡) $dX > \mathbf{L}$ is X, so by (13) and replacement we reach the contradiction

(14) $X \equiv \neg X$

The contradiction may then constructed as the conclusion of a formal argument.

However, we may also reach the same result via an intuitive argument along the lines of the standard antinomic reasoning. The argument relies on a tacit premise: that supposing A is the same as supposing that someone (e.g. I myself) says A, which is indeed plausible but depends on the meaning of the terms "supposing" and "saying".

(a) Let us suppose that X, i.e. $dX > \mathbf{L}$ is true. Thus we are supposing that I am saying $X(dX)$, so by Conditional Modus Ponens **L**, i.e. that I am saying something false. So by supposing that X is true I conclude that I am a liar in saying this. Contradiction.

(b) Let us suppose that X, i.e. $dX > \mathbf{L}$, is false. This implies implies supposing the truth of $dX > d\mathbf{T}$ by axiom LEM. Then I would tell the truth in saying that X is true. Contradiction.

§5. To conclude this note, we may show that it is possible to build in conditional terms two well-known variants of the Liar which are the Epimenides and the Truth Teller.

Epimenides the Cretan says

(15) All Cretans are liars.

In the conditional paraphrase we have proposed above (15) means:

(16) If Epimenides said that some Cretan tell the truth, Epimenides would be lying

Suppose Epimenides tells the truth. Then there is some Cretan (i.e. Epimenides) who tells the truth. So if Epimenids makes the assertion that some Cretan tells the truth he is telling the truth.Then by Conditional Modus Ponens applied to (16) it follows that he is lying. Contradiction.

Suppose Epimenides tells a falsehood. Then it is possible that, *ceteris paribus*, Epimenides says that some Cretan tell the truth and that Epimenides tells the truth in making this assertion. This is not a contradiction unless there is only one Cretan in the world, i.e. Epimenides.

So, given that the first alternative implies a contradiction and that Epimenides is not the only Cretan, Epimenides makes a false assertion.

The so-called Truth Teller is the statement:

(17) What I am saying is true or equivalently

(18) Xt: Xt is a true statement.

In conditional terms the rendering of (17) is

(19) I would be lying if I said that the present statement is false and the rendering of (18) is

(20) I would be lying if I said that Xt> is false

Suppose Xt> is true. Then the conditional (20) is also true. Suppose Xt> it is false: this means that it logically possible that Xt> is false and that I am telling the truth about this fact, which is the negation of (20).

Then what we may say about (20) is that it is true if and only if it is true, and that there is no procedure to establish if is true or false. (20) is then an undecidable statement, just as the standard Truth Teller.

References

ARLO-COSTA, H. The logic of conditionals. *Stanford Enc. Of Philosophy*, 2007. Disponível em: <http://stanford.library.usyd.edu.au/archives/sum2010/entries/logic-conditionals/>.

CHELLAS, B. Basic conditional logic. *Journal of Phil. Log.*, v. 4, p. 133–154, 1975.

ELDRIDGE-SMITH, P. The pinocchio paradox. *Analysis*, v. 70, p. 212–215, 2010.

MCCALL, S. Connexive logic. *Stanford Enc. Of Philosophy*, 2007. Disponível em: <http://http://plato.stanford.edu/entries/logic-connexive/>.

PRIEST, G. Intensional paradoxes. *Notre Dame Journal of Formal Logic*, v. 32, p. 193–211, 1991.

PRIOR, A. N. On a family of paradoxes. *Notre Dame Journal of Formal Logic*, v. 2, p. 16–32, 1961.